Semi-analytical approaches to gravitational radiation from astrophysical sources

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Adiabatic inspiral and heuristic view of the plunge[1]

- Geometrized units and Boyer-Linquist coordinates (t, r, θ, ϕ).
- Effective potential, $m \equiv$ TP mass, $M \equiv$ BH mass, $a = S/M$,

$$V_{\rm eff} = 1 - \frac{2M}{r} + \frac{[L^2/m^2 - a^2(E^2/m^2 - 1)]}{r^2} - \frac{2M(L/m - aE/m)^2}{r^3}$$

- Particle initially at a circular orbit

$$E_{\rm circ}/m = (r^2 - 2Mr + aM^{1/2}r^{1/2})/[r(r^2 - 3Mr + 2aM^{1/2}r^{1/2})]^{1/2}$$

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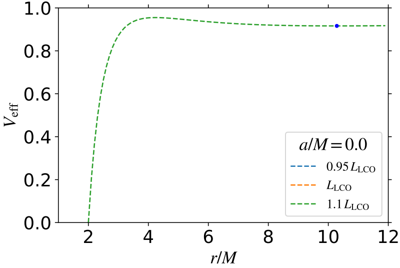

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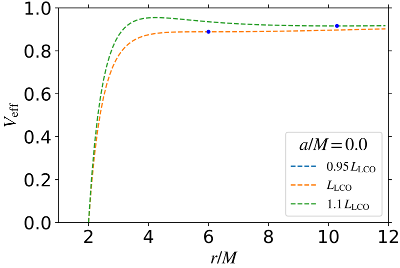

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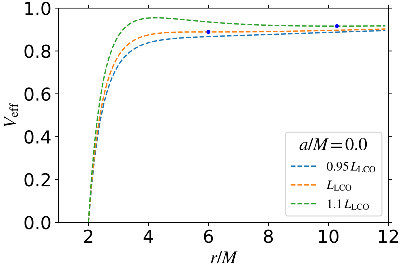

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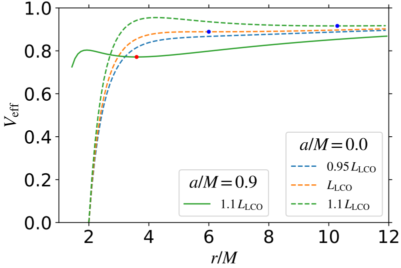

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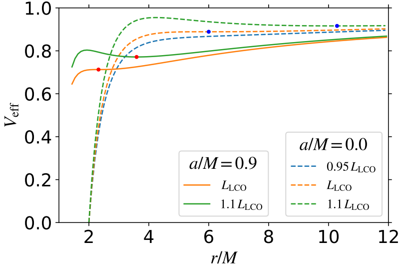

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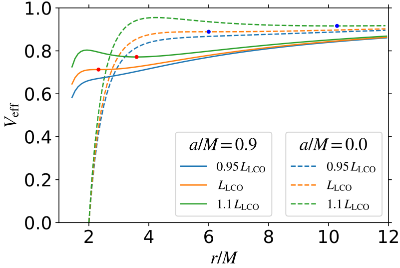

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Hamiltonian approach and helicoidal drifting sequence (HDS)[2]

- The Hamiltonian of the test particle of mass m in the field of the Kerr black hole of mass M is given by

$$H = -p_t = -N^i p_i + N\sqrt{m^2 + \gamma^{ij} p_i p_j}, \tag{1}$$

where $N = 1/\sqrt{-g^{00}}$, $N^i = -g^{ti}/g^{tt}$, $\gamma^{ij} = g^{ij} - g^{ti} g^{tj}/g^{tt}$.

- Dynamical equations are:

$$\dot{r} = \frac{\partial H}{\partial p_r} \qquad \Omega \equiv \dot{\phi} = \frac{\partial H}{\partial L}, \tag{2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} + \mathcal{F}_r^{\rm nc} \qquad \dot{p}_\phi = -\mathcal{F}_\phi^{\rm nc} \tag{3}$$

where $\mathcal{F}_r^{\rm nc}$ and $\mathcal{F}_\phi^{\rm nc}$ are the radial and azimuthal non-conservative radiation-reaction forces.

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- Dynamical equations are: Circular orbits

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- Particle falls due to the loss of energy and angular momentum, and explicit contribution of the conservative radial force.
- Adiabatic parameter $\eta \equiv \dot{\Omega}/\Omega^2 \approx \dot{r}/(r\Omega) \ll 1$.
- All dynamical variables are finite at LCO

[2] Rodriguez, Rueda & Ruffini, **Astronomy Reports,** 62, 940

How to calculate $\dot{E}_{\mathrm{GW}}$ and $\dot{L}_{\mathrm{GW}}$? GWR from circular Orbits on Kerr Spacetime

- Test particle in a stable circular orbit at r and co-rotating with the BH

$$\Omega_{\mathrm{corot}} = \frac{M^{1/2}}{(r^{3/2} + aM^{1/2})} \tag{4}$$

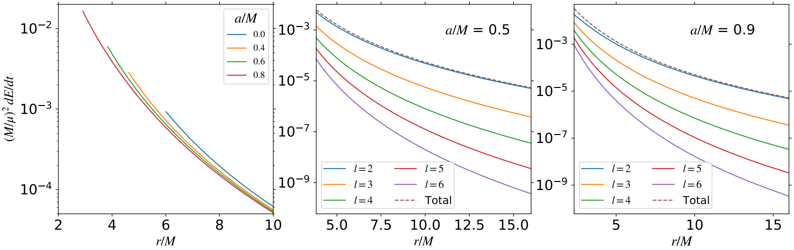
- The energy and angular momentum fluxes to ∞ are given in terms of $Z^H_{lm\omega}$, which are obtained from BH perturbation theory [Teukolsky, **ApJ**, 185, 635 (1973)], namely by solving the Sasaki-Nakamura Eq. [Sasaki & Nakamura, **Phys. Lett. A**, 86, 68 (1982)]:

$$\frac{dE}{dt} = \sum_{l,m}^{\infty} \frac{|\tilde{Z}^H_{lm\omega}|^2}{4\pi\omega_m^2}, \tag{5}$$

$$\frac{dL}{dt} = \sum_{l,m}^{\infty} \frac{m|\tilde{Z}^H_{lm\omega}|^2}{4\pi\omega_m^3}, \tag{6}$$

- The GW angular frequency for each multipole $\omega_m = m\Omega$
- NO RADIATION FROM CIRCULAR ORBITS INSIDE THE LCO.

Energy Flux



Energy fluxes to infinite test particle in a circular orbit. Rodriguez, Rueda & Ruffini, **Astronomy Reports,** 62, 940

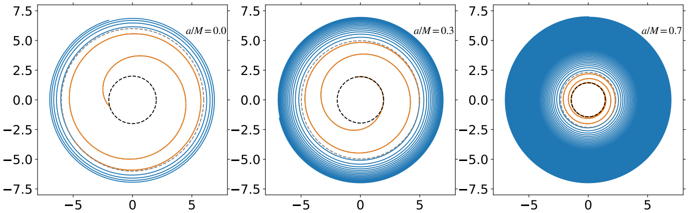
- For quasi-circular, adiabatic motion we have:

$$\mathcal{F}_r^{\rm nc} = 0, \tag{7}$$

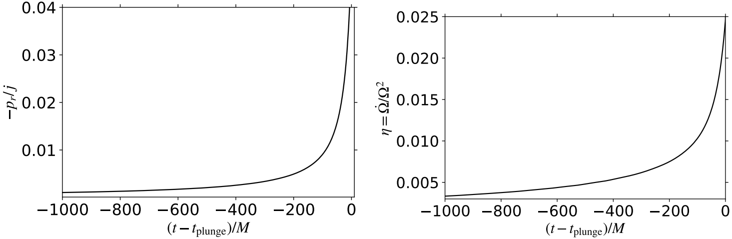
$$\mathcal{F}_\phi^{\rm nc} = -\frac{1}{\Omega}\frac{dE}{dt}. \tag{8}$$

Test particle dynamics in HDS approach

- Test particle trajectories $\nu = mM/(m+M)^2 = 0.01$, for different a/M:



- The dynamics includes the contribution of the radial momentum which is not negligible near the LCO.
- $t_{\rm plunge}$ is defined as the time when the particle crosses the LCO **in the HDS.**



Left: ratio $-p_r/(p_\phi/m) = p_r/j$ for $a/M = 0.9$. Right: Adiabatic parameter. Rodriguez, Rueda, & Ruffini, Astronomy Reports, 62, 940 (2018)

Transition to plunge: comparison with other approaches

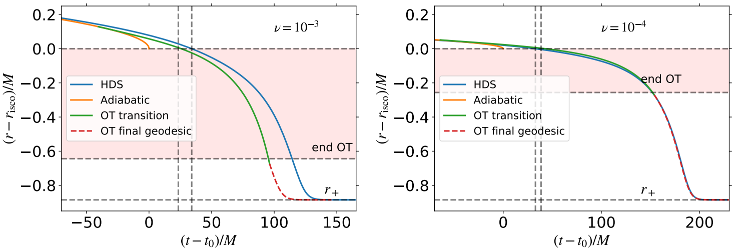
- Taylor expansion of $V_{\rm eff}$ around the LCO[3]:

$$\ddot{r} = \partial_r^3 F|_{\rm LCO}(r - r_{\rm LCO})^2/4 + \partial_{rE}F|_{\rm LCO}(E - E^{\rm circ}_{\rm LCO})/2 + \partial_{rL}F|_{\rm LCO}(L - L^{\rm circ}_{\rm LCO})/2,$$

where, $F := r^4 \frac{V_{\rm eff}}{V_t^2}$, $\quad V_t = \frac{aL}{m} + \frac{r^2+a^2}{m(r^2-2Mr+a^2)}[E(r^2+a^2) - La]$

- Energy and angular momentum losses evaluated at LCO ($r_{\rm adiab}(t_0) = r_{\rm LCO},\ t_0 \neq t_{\rm plunge}$)

$$(E - E^{\rm circ}_{\rm LCO}) = (t - t_0)\dot{E}_{\rm GW}|_{\rm LCO},\ (L - L^{\rm circ}_{\rm LCO}) = (t - t_0)\Omega^{-1}\dot{E}_{\rm GW}|_{\rm LCO}.$$



Comparison $a/M = 0.9$. See Rodriguez, J. F., Rueda, J. A, and Ruffini, R, Astr. Rep., 2018, 62, 940

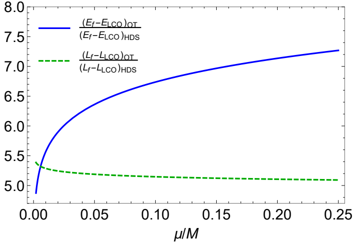
- GW radiation at the same rate as the LCO, long after the test particle has passed it.

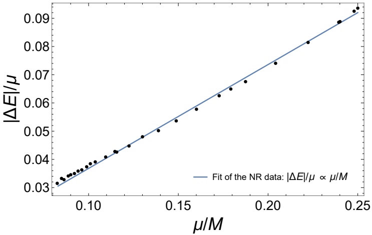
[3] Ori & Thorne, **Phys. Rev. D**, 62, 124022 (2000); Sundararajan, **Phys. Rev. D**, 77, 124050 (2008)

- The energy "deficits" at the end:

$$\Delta E_{\rm def} = E^{\rm circ}_{\rm LCO} - E^{\rm formalism}_{\rm plunge,f} \tag{9}$$

- $E^{\rm HDS}_{\rm plunge,f} := H(t_{\rm plunge}) < E^{\rm circ}_{\rm LCO}$





Our work: Rodriguez, J. F., Rueda, J. A, and Ruffini, R, Astronomy Reports, 2018, Vol. 62, No. 12, pp. 940–952. Comparison with Ori & Thorne (2000) (left) and NR from SXS catalog `https://www.black-holes.org/` (right)

- In the HDS there is less radiated energy (angular momentum) than in the results of Ori & Thorne (2000).

Semi-analytic approach to binary black holes (comparable masses)

- Motivation: work of Aninos et al. (1995) (among other works), also as a "working hypothesis" $\rightsquigarrow$ Newtonian center-of-mass point of view

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Principles of "test particle waveform" formalism

1. Test particle orbits on **Kerr spacetime**: HDS approach
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"Test particle" waveform construction

1. The gravitational waveform can be constructed from "circularized waves":

$$\frac{1}{2}\left(h_+ - ih_\times\right) = -\frac{1}{R}\sum_{l,m}\frac{Z^H_{lm\omega}}{\omega_m^2}\,{}_{-2}S_{lm}(\Theta)e^{im\Phi}e^{-i\omega_m(t-R^*)}, \tag{10}$$

   where R is the distance to the observer, Θ is the angle between the axis of rotation and the observer, Φ is the azimuthal coordinate of the orbiting body at $t = 0$; R^* is the Kerr "tortoise".
2. The complex number $Z_{lm\omega}$ evolves with time, inducing a variable wave amplitude and phase shift.

$$(r(t), \phi(t)) \quad \text{“test particle” trajectory coordinates}$$

,

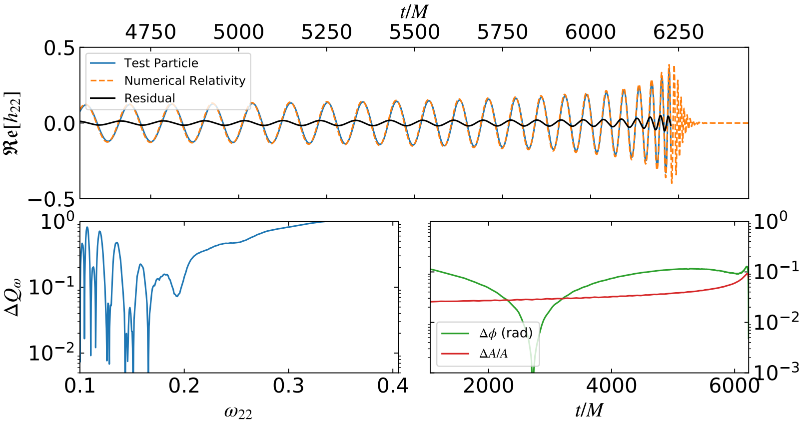
$$\omega_m(t - R^*) \mapsto m\phi(t - R^*).$$

"Test particle" and NR waveforms

Intrinsic phase-time parameter $Q_\omega = \omega^2/\dot{\omega}$

Working hypothesis

HDS + Newtonian center-of-mass



Numerical-relativity waveform `BBH:0230`, $m_1 = m_2$ and initial spins $a_1/m_1 = a_2/m_2 = 0.8$, final Kerr BH spin $a_f/M_f = 0.907516$. Intrinsic time-domain phase difference evolution $\Delta Q_\omega = |Q_\omega^{\mathrm{TP}} - Q_\omega^{\mathrm{NR}}|$

.

Comparison with NR

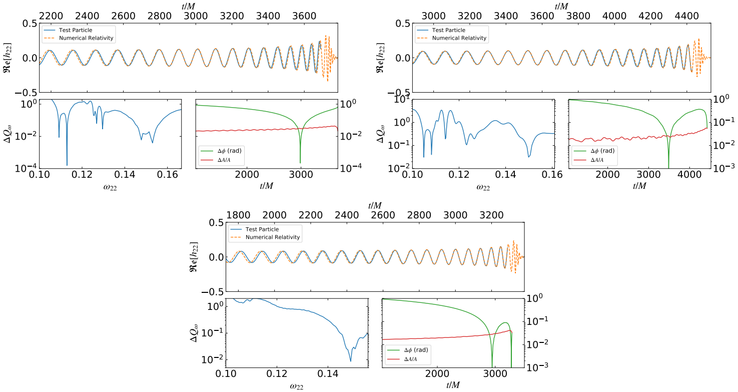
- BBH numerical relativity simulations with equal masses components, $m_1/m_2 = q$. Initial components can have equally aligned spins $a_1/m_1 = a_2/m_2$. We studied the special case without spin (two Schwarzschild BHs).
- Fitting factor F,

$$F \equiv (h_1|h_2)/\sqrt{(h_1|h_1)(h_2|h_2)}, \quad (h_1|h_2) \equiv 4\mathfrak{Re}\left[\int_0^\infty h_1(f)\tilde{h}_2(f)/S_n(f)df\right],$$

Simulation	a_i/m_i	a_f/M_f	$a_{\rm eff}/M_f$	F
`BBH:0001`	1.209309×10^{-7}	0.686461	0.36	0.96
`BBH:0157`	0.949586	0.940851	0.99	0.93
`BBH:0228`	0.600000	0.857813	0.80	0.972
`BBH:0230`	0.800000	0.907516	0.9075	0.993

Rodriguez, J. F., Rueda, J. A. and Ruffini, R, **JCAP**, 2018, 030

- BBH simulations with spin-less initial components and different mass ratios $q < 1$



SXS spinless BBH simulations `https://www.black-holes.org/`, BBH:0169, $q = 1/2,\ a_{\rm eff} = 0.33$; BBH:0169, $q = 1/3,\ a_{\rm eff} = 0.329$; BBH:0169, $q = 1/4,\ a_{\rm eff} = 0.25$.
Rodriguez, J. F., Rueda, J. A. and Ruffini, R, **JCAP**, 2018, 030

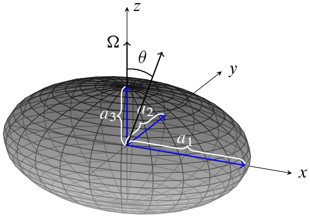
“FRAME DRAGGING” DUE TO THE ORBITAL ANGULAR MOMENTUM

Incompressible ellipsoidal figures of equilibrium

- Classical work of Chandrashekar, *Ellipsoidal Figures of Equilibrium* (1969)
- Evolution of incompressible self-gravitating object driven by GW reaction $\Longrightarrow$ conserved quantity: circulation along the equator. Bonnie, D. Miller, The Astrophysical Journal, 187, 606, (1974).
- Evolution along Riemann S-type (self-gravitating object supported by rotation and internal motions) sequence with constant circulation C,

$$C = \pi a_1 a_2 \big(\zeta + 2\Omega\big),$$

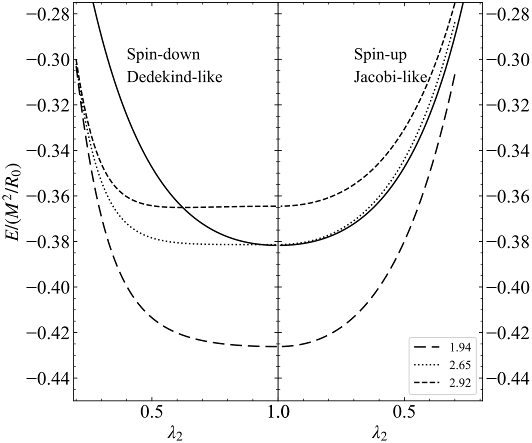
where Ω spin and ζ vorticity, and they are parallel.



A compressible ellipsoid characterized by the polytropic index n, $\lambda_2 := a_2/a_1$, $\lambda_3 := a_3/a_1$, Ω and vorticity ζ

Compressible models: polytropic EOS

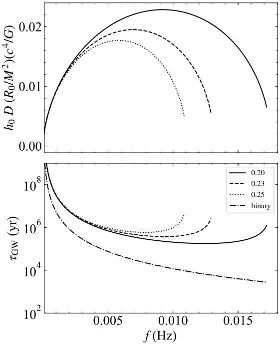
- Generalization to objects whose internal matter is described by a polytropic EOS [Lai, Rasio & Shapiro, **ApJ S**, 88, 205 (1993)].
- Time unit $\tau_{\rm CEL} = (\pi G \bar{\rho}_0)^{-1/2}$, where $\bar{\rho}_0 = M/(4/3\pi R_0^3)$, R_0 := radius of the non-rotating polytrope (same *n*) with the same mass.
- Evolution *a la Landau-Lifshitz*



Quasi-static evolution of a CEL with $n = 1$ and different circulations $C/\pi = 1.95, 2.39, 2.92$

Polytropic index $n \to 3$: chirping ellipsoid (CEL)

- New phenomena: Early period increasing GW frequency and amplitude
- typical frequency is ~ 1 mHz. Space-based interferometers: LISA, TianQin.



GW strain and timescale of a CEL with $n = 2.95$, $M = 1.0\, M_\odot$ which is the same of the non-rotating spherical star with radius $R \approx 6000$ km. The value of the circulation for the continuous, dashed and dotted line are $\mathscr{C} = (\kappa_n MC/5\pi)/(GM^3 R_0)^{1/2} = (0.20, 0.23, 0.25)$, respectively. Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, JCAP 2023.

Intrinsic GW phase-time evolution of CELs

- CELs are quasi-monochromatic
- Intrinsic phase-time parameter $Q_\omega \equiv \frac{\omega^2}{\dot{\omega}} = \frac{d\phi}{d\ln\omega} = 2\pi \frac{dN}{d\ln f}$

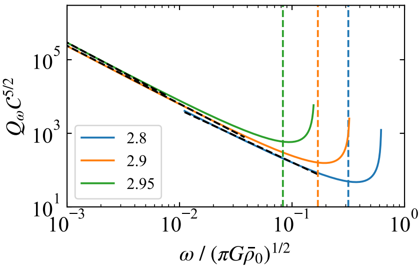


Figure 1: Intrinsic phase-time of CELs with $n = 2.8, 2.9, 2.95$. Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, **JCAP 2023**.

- Intrinsic phase empirical fits,

$$Q_\omega^{\rm CEL} \approx \frac{\mathcal{A}_n}{\mathcal{C}^{5/2}} \left[\frac{\omega}{\sqrt{\pi G \bar{\rho}_0}} \right]^\alpha \quad \mathcal{C} = G M_{\rm CEL}/(c^2 R_0)$$

- Intrinsic phase-time evolution of Newtonian binaries

$$Q_\omega^{\rm bin} = \frac{5}{3} 2^{-7/3} \left(\omega M_{\rm chirp}\right)^{-5/3}$$

CELs parameters

- Polytropic structure constants $(n, \kappa_n, k_1, k_2, k_3)$ and the Q_ω power-law empirical fitting parameters.

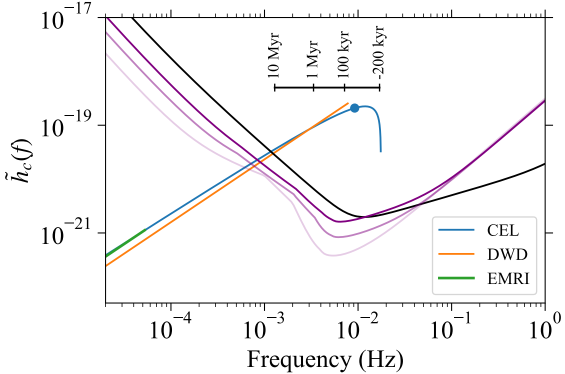
n	κ_n	k_1	k_2	k_3	$\mathcal{A}_n$	α
1.0	0.65345	0.5	0.81289	2.2472	2.68	-1.081
2.0	0.38712	1.1078	0.71618	1.6562	4.003	-1.222
2.5	0.27951	1.4295	0.67623	1.4202	4.060	-1.447
2.7	0.24109	1.55971	0.66110	1.33194	5.926	-1.365
2.9	0.20530	1.69038	0.64630	1.24621	4.940	-1.571
2.95	0.19676	1.72309	0.64265	1.22511	4.369	-1.614
2.97	0.19340	1.73617	0.64119	1.21669	3.760	-1.640
2.99	0.19005	1.74925	0.63973	1.20829	3.817	-1.652

- In the limit $n \to 3$, $\alpha \to -5/3$: CEL-binary equivalence,

$$Q_\omega^{\rm CEL} = Q_\omega^{\rm bin} \Longleftrightarrow \frac{M_{\rm CEL}}{M_{\rm chirp}} = 2^{7/5}\left(\frac{3A_3}{5}\right)^{3/5}\sqrt{3/4}$$

CELs detection

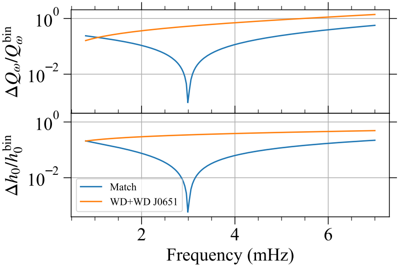
$$\tilde{h}_c(f) = h_0(f)\sqrt{N} = h_0(f)\sqrt{fT_{\rm obs}},$$



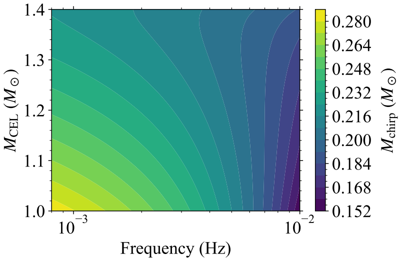
Reduced characteristic amplitude, $\tilde{h}_c$, of a CEL, a double WD (DWD) and an EMRI. The CEL has a mass $M_{\rm CEL} = 1.0\, M_\odot$ and compactness $\mathcal{C} \approx 2.5 \times 10^{-4}$ (blue), according to the relativistic Feynman-Metropolis-Teller EOS. The polytropic index is $n = 2.95$, and is located at a distance $D = 1$ kpc. The observing time has been set to $T_{\rm obs} = 2$ yr.. Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, JCAP 2023

CEL-binary degeneration

Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, JCAP 2023



CEL ($n = 2.95$) with $M_{\rm CEL} = 1.0\, M_\odot$ and $\mathcal{C} = 2.5 \times 10^{-4}$, and an equivalent binary system with $M_{\rm bin} = 0.24\, M_\odot$. Also comparison of the CEL signal with that of the detached binary DWD J0651 ($M_{\rm chirp} = 0.31\, M_\odot$).



Contours of constant chirp mass of the equivalent binary as a function of the CEL mass and the observed frequency. In general, the chirp mass of the equivalent binary depends on the $\mathcal{C}$, $M_{\rm CEL}$, and on the observed frequency f. However, once the EOS is selected, the mass-radius relation is fixed implying that $M_{\rm chirp}$ depends only on $M_{\rm CEL}$ and f.

CEL-binary degeneration

- Phase difference during 1 yr,

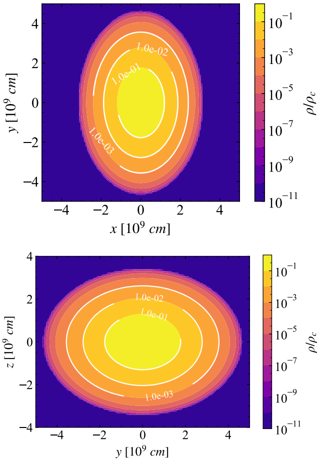
$$\Delta\phi_{1y} = \int_{\omega_0}^{\omega_{1yr}} \Delta Q_\omega d \ln \omega,$$

  where $\Delta Q_\omega = |Q_\omega^{\rm CEL} - Q_\omega^{\rm bin}|$
- End for EMRIs: $f_{\rm td} \approx (Gm_2/R_2^3)^{1/2}/(2.4^{3/2}\pi)$,
- End for DWD: frequency when one of the components fills its Roche-lobe.
- CEL-binary equivalence parameters, Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, JCAP 2023

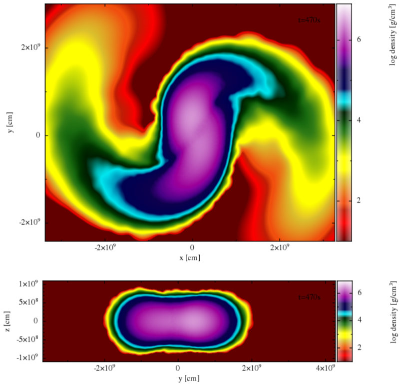
$M_{\rm CEL}$ $(M_\odot)$	$\mathcal{C}$ (10^{-4})	$f_{\rm end}^{\rm CEL}$ (mHz)	$M_{\rm chirp}$ $(M_\odot)$	m_1 $(M_\odot)$	m_2 $(M_\odot)$	$f_{\rm end}^{\rm bin}$ (mHz)	Type-like	f_0 (mHz)	$\Delta\phi_{1y}$	$\frac{\Delta h_0}{h_0}\big\|_{1y}$	$\frac{D_{\rm CEL}}{D_{\rm bin}}$	SNR
1.0	2.5	9.20	0.32	1940.62	0.0001	0.053	EMRI	0.05	3.631×10^{-10}	5.937×10^{-13}	0.778	und.
			0.28	0.35	0.30	13.38	PG1101+364	1.0	5.004×10^{-5}	2.515×10^{-9}	0.773	0.687
			0.24	0.45	0.18	7.76	J0106-1003	3.0	5.018×10^{-3}	6.455×10^{-8}	0.835	9.079
1.4	20.0	148.70	0.48	2916.81	0.0015	0.064	EMRI	0.05	5.521×10^{-10}	9.322×10^{-13}	0.808	und.
			0.45	0.59	0.45	19.92	WD0028-474	1.0	3.868×10^{-5}	3.106×10^{-9}	0.776	1.511
			0.43	0.52	0.47	21.30	WD0135-052	3.0	2.660×10^{-3}	6.344×10^{-8}	0.766	23.88
			0.42	0.51	0.45	20.25	WD1204-450	6.0	4.148×10^{-2}	4.377×10^{-7}	0.763	119.89
			0.41	0.47	0.47	21.48	WD1704-481[4]	9.0	2.135×10^{-1}	1.375×10^{-6}	0.764	145.73

[4] Same chirp mass

CEL: aftermath of binary white dwarf mergers



Isodensity curves of a CEL with polytropic index $n = 2.95$, $M_{\rm CEL} = 1.2\,M_\odot$, and central density $\rho_c = 1.2 \times 10^8$ g cm^{-3} rotating with angular velocity $\Omega/\sqrt{\pi G \bar{\rho}_0} = 0.02$. All the curves are *self-similar* to the ellipsoid with axes ratio $a2/a_1 = 0.68$ and $a_3/a_1 = 0.77$.



Density map of a section in the orbital plane (top panel) and in the polar plane (bottom panel) of a $0.6 + 0.6\,M_\odot$ DWD merger simulated with 5×10^4 SPH particles. This snapshot is taken 9 orbital periods after mass transfer begins.

Rates (Rodriguez, Rueda, Ruffini, Zuluaga, Iglesias-Blanco and Loren-Aguilar, JCAP 2023)

- Globular cluster $\rightarrow$ star with planetary companion $\rightarrow$ migration to the center and capture EMRI rate: $0.02 - 0.5\ \mathrm{yr}^{-1}$
- DWD, systems that will merger within Hubble time, rate $0.0064 - 0.512\ \mathrm{yr}^{-1}$
- CEL result of binary white mergers (no supernova) $0.0056 - 0.45\ \mathrm{yr}^{-1}$

BNS merger (Juan Diego's Talk)

- BNS produced by a BdHN II, L. Becerra et al Universe 2023

	m $[M_\odot]$	j	Ω $[\mathrm{s}^{-1}]$	R_{eq} [km]	I [g cm^2]	Ω $[\mathrm{s}^{-1}]$	R_{eq} [km]	I [g cm^2]
				GM1 EOS			TM1 EOS	
νNS	1.505	0.259	1114.6	14.03	2.04×10^{45}	1077.1	14.47	2.11×10^{45}
NS	1.404	-0.011	-52.14	14.01	1.85×10^{45}	-56.6	14.49	1.93×10^{45}

- Conservation of baryonic mass

$$M_b = m_{b,c} + m_{\mathrm{ej}} + m_d, \quad M_b = m_{b,1} + m_{b,2}.$$

- The relation among its baryonic mass, $m_{b,i}$, gravitational mass, m_i, and angular momentum J_i is,

$$\frac{m_{b,i}}{M_\odot} \approx \frac{m_i}{M_\odot} + \frac{13}{200}\left(\frac{m_i}{M_\odot}\right)^2\left(1 - \frac{1}{130} j_i^{1.7}\right), \quad i = 1, 2, c,$$

- Conservation of angular momentum

$$J_{\mathrm{merger}} = J_c + J_d + \Delta J,$$

Remnant NS, maximal disk mass and no disk (Juan Diego's Talk)

- Conservation of mass-energy

$$E_{\rm GW} + E_{\rm other} = \Delta M c^2 = [M - (m_c + m_{\rm ej} + m_d)]c^2, \tag{11}$$

- Limiting case with $\Delta J = 0$, which corresponds to the case with maximum disk mass, $m_c = 2.697\, M_\odot$, $m_d = 0.073\, M_\odot$.

$$E_{\rm other} = \Delta M c^2 = [M - (m_c + m_{\rm ej} + m_d)]c^2 \approx (M - m_c - m_d)c^2 \tag{12}$$

$$\approx 0.139\, M_\odot c^2 \approx 2.484 \times 10^{53} \quad \text{erg} \tag{13}$$

- Limiting case with $m_d = 0$ which corresponds to the maximum angular momentum loss. $\Delta J = 0.331\, GM_\odot^2/c$, and the maximum remnant's mass, $m_c = 2.756\, M_\odot$, $E_{\rm GW}^{\rm pm} \approx 0.0079\, M_\odot c^2 \approx 1.404 \times 10^{52}$ erg.

$$\Delta M c^2 = [M - (m_c + m_{\rm ej})]c^2 \approx (M - m_c)c^2 \tag{14}$$

$$\approx 0.153\, M_\odot c^2 \approx 2.734 \times 10^{53} \quad \text{erg} \tag{15}$$

Unipolar model WD-WD (Caravalho et Apj 940, 2022)

- Homopolar model
- Beyond pure gravitational quadrupole emission

$$\frac{\dot{\omega}_0}{\omega_0} = -\frac{\dot{P}}{P} = \frac{1}{g(\omega_0)}\left[\dot{E}_{\rm GW} - \frac{L}{1-\alpha}\right], \qquad (16)$$

$$\frac{\dot{\alpha}}{\alpha} = -\frac{1}{g(\omega_0)}\left\{\dot{E}_{\rm GW} - \frac{L}{1-\alpha}\left[1 + \frac{g(\omega_0)}{\alpha I_1 \omega_0^2}\right]\right\}, \qquad (17)$$

$$\frac{L}{1-\alpha} = 7.72\times 10^{32}\left(\frac{\tilde{B}}{10^6\ {\rm G}}\right)^2\left(\frac{R_1}{10^9\ {\rm cm}}\right)^6 \times \left(\frac{R_2}{10^9\ {\rm cm}}\right)^2\left(\frac{M_\odot}{M}\right)^{4/3}\left(\frac{100\ {\rm s}}{P}\right)^{14/3} {\rm erg\ s}^{-1}, \qquad (18)$$

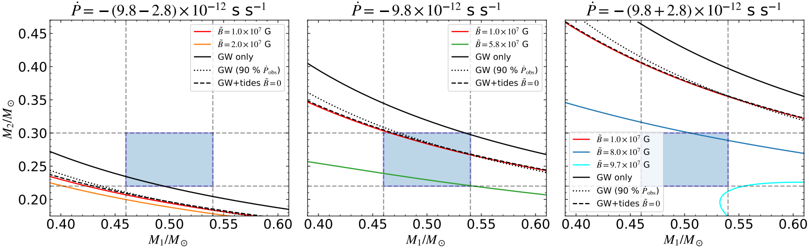


Figure 2: Constraints for SDSS J0651+2844.

Thank you